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Two-dimensional piezoelectricity. Part II: general solution, Green's function and interface cracks

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Abstract

The complete solution space of a piezoelectric material is the direct sum of several orthogonal eigenspaces, one for each *distinct* eigenvalue. Each one of the 14 different classes of piezoelectric materials has a distinct form of the general solution, expressed in terms of the eigenvectors of the zeroth and higher orders and a kernel matrix containing analytic functions. When these functions are chosen to be logarithmic, one obtains, in a unified way, Green's function of the infinite space as a single 8×8 matrix function $\mathbf{G}_\infty$ for the various load cases of concentrated line forces, dislocations, and a line charge. This expression of Green's function is valid for all classes of nondegenerate and degenerate materials. With an appropriate choice of the parameters, it reduces to the solution of a half space with concentrated (line) forces at a boundary point, and with dislocations in the displacements. As another application, eigenvalues and eigensolutions are obtained for the bimaterial interface crack problem.

Keywords: Piezoelectricity; Anisotropic elasticity; Eigensolutions; Green's function; Interface cracks

1. Introduction

The analysis of piezoelectricity in Part I of this paper yields 11 distinct types of eigenvalues. They depend on the multiplicity, and on whether the eigenvalues are normal, abnormal, or superabnormal. A p-tuple eigenvalue determines a p-dimensional eigenspace containing p eigenvectors of the various orders. Taking all possible combinations of the different types of eigenvalues with $Im[\mu] > 0$, and with a total multiplicity equal to 4, one obtains 14 distinct classes of piezoelectric materials, including four that

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are nondegenerate. For each class, the base matrix $\mathbf{Z}$ is nonsingular, i.e., consisting of eight independent eigenvectors of the zeroth and higher orders. General solutions of all classes of materials are given here in a concise, unified manner, in terms of $\mathbf{Z}$ and a kernel matrix which contains four arbitrary analytic functions.

The algebraic structure of the solution space of linear piezoelectricity is formally analogous to those of the related problems of anisotropic elasticity (without piezoelectric effects) and unsymmetric anisotropic plate theory, although the dimensions and properties of the solution spaces are generally different in different theories. One has the same formal expressions of the pseudometrics, the projection operators into the various eigenspaces, and of the intrinsic tensors, but some intrinsic tensors cease to be positive definite in piezoelectricity unless properly modified.

Choosing the analytic functions in the kernel matrix to be the logarithmic function, one obtains Green's function of the infinite space with line loads, dislocations and line charge at the origin. Analytical solutions are also obtained for a half space under the same loads applied at a boundary point, and for a bimaterial interface crack, also regardless of material degeneracy. Solutions to other problems, such as Green's functions of a half space with various types of boundary conditions, and of an infinite space with an elliptical hole, may be obtained in a manner similar to the corresponding problems in 2-D elasticity (Yin, 2004) and unsymmetric laminated plate theory (Yin, in press a, b).

2. Orthogonality and separability of eigenspaces

2.1. Orthogonality

The analysis of Part I shows that all eigensolutions of the various orders have the form described by Theorem 4. Differences in the type of eigenvalues, i.e., normal, abnormal and superabnormal, merely affect the choice of the function $\boldsymbol{\eta}(\mu)$, which may be a column of $\mathbf{W}(\mu)$, or of $\mathbf{I}_3$, or one of the vector functions in Eq. (I-3.10) (Notice that here and in the following, when referring to equations in Part I of this paper, the equation numbers are preceded by the prefix I-≅). Let $\boldsymbol{\xi} = (\mathbf{J}\boldsymbol{\eta})^{(k)}(\mu)$ and $\boldsymbol{\xi}' = (\mathbf{J}\boldsymbol{\eta})^{(l)}(\mu')$ be eigensolutions associated with two distinct eigenvalues μ and μ'. Then the following two set of equations, when evaluated respectively at μ and μ',

$$\mathbf{M}\boldsymbol{\eta} = \ldots = (\mathbf{M}\boldsymbol{\eta})^{(k)} = 0, \quad \mathbf{M}\boldsymbol{\eta} = \ldots = (\mathbf{M}\boldsymbol{\eta})^{(l)} = 0, \tag{2.1a,b}$$

are valid provided that the order of differentiation k is smaller than the multiplicity of μ and l is smaller than the multiplicity of μ'. Furthermore,

$$[\![\boldsymbol{\xi},\boldsymbol{\xi}']\!] = \sum_{0\leqslant p\leqslant k}\ \sum_{0\leqslant q\leqslant l} (k,p)(l,q)\left\{\boldsymbol{\eta}^{(k-p)}(\mu)\right\}^{\mathrm{T}}[\![\mathbf{J}^{(p)}(\mu),\mathbf{J}^{(q)}(\mu')]\!]\boldsymbol{\eta}^{(l-q)}(\mu').$$

Substituting Eq. (I-4.3) into the last equation, and using (2.1a,b), one obtains $[\![\xi,\xi']\!] = 0$. This establishes the orthogonality of the eigenvectors associated with distinct eigenvalues:

Theorem 1. *If ξ and ξ' are eigenvectors associated with two distinct eigenvalues, then, irrespective of the order and the type of each eigenvalue, one has $[\![\xi,\xi']\!] = 0$. Hence the eight-dimensional complex vector space is the direct sum of a number of orthogonal eigenspaces, one for each distinct eigenvalue.*

For each zeroth-order or high-order eigenvector of a multiple eigenvalue μ_0, the complex conjugate vector is an eigenvector of the conjugate eigenvalue $\bar{\mu}_0$. While a zeroth-order solution is characterized completely by a single eigenvector, $\boldsymbol{\chi} = f\boldsymbol{\xi}$, higher-order eigensolutions generally involve all eigenvectors of the same and lower orders belonging to that eigenspace.

2.2. Separability of the eigenspaces of abnormal and superabnormal eigenvalues

Notice that the pseudometrics of Eqs. (I-5.11a,b) and (I-5.16a) are block-diagonal matrices. Therefore, for every abnormal eigenvalue μ_0 of multiplicity p considered in Theorems 8 and 9 of Part I, the eigenspace is the direct sum of two orthogonal subspaces of dimensions p_1 and $p_2 = p - p_1$ possessing, respectively, eigensolutions of increasing orders varying from 0 to $p_1 - 1$ and from 0 to $p_2 - 1$. Orthogonality implies that the eigensolutions of the two groups are uncoupled. These two subspaces of the eigenspace are effectively two separate eigenspaces that happen to have the same eigenvalue.

The pseudometric of a double abnormal eigenvalue as given by Eq. (I-5.4a) may be reduced to a diagonal matrix by using the similarity transformation

$$\boldsymbol{\tau} = \begin{bmatrix} 1 & W'_{13}/W'_{33} \\ 0 & 1 \end{bmatrix}$$

For a triple superabnormal eigenvalue, there are three independent eigenvectors of the zeroth order. The pseudometric of Eq. (I-5.17) may be reduced to a diagonal matrix by using an appropriate 3×3 similarity transformation $\boldsymbol{\tau}$, and the transformed eigenvectors are still of the zeroth order. For a quadruple superabnormal eigenvalue, the similarity transformation also reduces the 3×3 submatrix $\mathbf{M}'(\mu_0)$ in Eq. (I-5.19) to a diagonal matrix, and the three zeroth-order eigenvectors remain zeroth-order under the transformation. The fourth eigenvector, of the first order, is unchanged under $\boldsymbol{\tau}$. Hence the new pseudometric is block diagonal, whose only nonzero elements are ω_{11}, ω_{44}, $\omega_{14} = \omega_{41}$, ω_{22} and ω_{33}. One has the following theorem:

Theorem 2. *The eigenspace of an abnormal-α eigenvalue of multiplicity p is separated into two orthogonal subspaces of dimensions 1 and $p - 1$. The eigenspace of an abnormal-β eigenvalue is separated into two orthogonal subspaces of dimensions 2 and $p - 2$. The eigenspace of a superabnormal eigenvalue is separated into three orthogonal subspaces of dimensions 1, 1 and $p - 2$. The eigenspace of a normal eigenvalue is irreducible.*

Notice that in the abnormal and superabnormal case, the similarity transformation $\boldsymbol{\tau}$ replaces a higher-order eigenvector by a linear combination of that eigenvector with the *lower-order* eigenvectors. The lower order eigenvectors remain unchanged in order, so that the hierarchy of orders of all eigensolutions in the eigenspace is also unchanged as the pseudometric is reduced to the block-diagonal form by the transformation $\boldsymbol{\tau}$. Such a similarity transformation cannot be found to decompose the pseudometrics of multiple *normal* eigenvalues as given by Eqs. (I-4.18b,c,d). Hence the eigenspace of a multiple normal eigenvalue is irreducible.

Let $\mathbf{X}$ be the p-dimensional eigenspace of a normal eigenvalue, or a p-dimensional *irreducible subspace* in the eigenspace of an abnormal or superabnormal eigenvalue. Then $\mathbf{X}$ contains p independent eigenvectors $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_p$ of the orders $0, \ldots, p - 1$, respectively. The k th order eigensolution $\boldsymbol{\chi}_{k+1}$ is a linear combination of the eigenvectors $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_k$ and, according to Eqs. (I-4.14b), (I-5.8b), (I-5.9b), (I-5.13b), (I-5.15b), none of the coefficients of the lower-order eigenvectors in this combination may vanish unless the analytic function f is a polynomial function of degree equal to or lower than k. Hence every higher-order eigensolution intrinsically involves the modes of lower-order eigenvectors that belong to the same irreducible subspace. The relation is intrinsic, because it is determined by the material, and independent of the external loads or boundary conditions.

3. Mathematical structure of the solution space

Let $\{\mu\}_{\perp}$ denote the sequence of *distinct* eigenvalues with positive imaginary parts, and let $\{\mu\}$ be the complete sequence obtained by joining $\{\mu\}_{\perp}$ and its complex conjugate sequence $\{\bar{\mu}\}_{\perp}$. Then each element

μ_k or $\bar{\mu}_k$ of $\{\mu\}$ is associated with an eigenspace $\mathbf{X}_k$ or $\overline{\mathbf{X}}_k$. We also use the same symbol $\mathbf{X}_k$ to denote the matrix composed of the eigenvectors in the space $\mathbf{X}_k$. Then

$$[\![\mathbf{X}_k,\mathbf{X}_j]\!] = 0 \quad \text{if } k \neq j \quad \text{and} \quad [\![\mathbf{X}_k,\overline{\mathbf{X}}_j]\!] = 0 \quad \text{for arbitrary } k \text{ and } j, \tag{3.1}$$

where

$$[\![\mathbf{X}_k,\mathbf{X}_j]\!] \equiv \mathbf{X}_k^{\mathrm{T}}\mathbf{II}\mathbf{X}_j, \quad \mathbf{II} \equiv \begin{bmatrix} \mathbf{0}_{4\times4} & \mathbf{I}_4 \\ \mathbf{I}_4 & \mathbf{0}_{4\times4} \end{bmatrix},$$

i.e., the eigenspaces are all mutually orthogonal. It was shown in Part I that the pseudometric $\boldsymbol{\omega}_k = [\![\mathbf{X}_k, X_k]\!]$ is a nonsingular matrix for every type of eigenvalue. Let $\mathbf{Z}_\perp$ be the 8×4 matrix obtained by joining all matrices $\mathbf{X}_k$, and let

$$\mathbf{Z} \equiv \left[\mathbf{Z}_\perp,\overline{\mathbf{Z}}_\perp\right], \quad \boldsymbol{\Omega}_\perp \equiv [\![\mathbf{Z}_\perp,\mathbf{Z}_\perp]\!] = <\boldsymbol{\omega}_k>, \quad \boldsymbol{\Omega} \equiv [\![\mathbf{Z},\mathbf{Z}]\!] = \mathbf{Z}^{\mathrm{T}}\mathbf{II}\mathbf{Z} = <\boldsymbol{\Omega}_\perp,\bar{\boldsymbol{\Omega}}_\perp>, \tag{3.2a,b,c}$$

where $<\boldsymbol{\omega}_k>$ stands for the block-diagonal matrix with the submatrices $\boldsymbol{\omega}_k$ as the diagonal blocks, and $<\boldsymbol{\Omega}_\perp,\bar{\boldsymbol{\Omega}}_\perp>$ is composed of two diagonal blocks $\boldsymbol{\Omega}_\perp$ and $\bar{\boldsymbol{\Omega}}_\perp$. Then

$$\mathrm{Det}[\boldsymbol{\Omega}] = |\,\mathrm{Det}[\boldsymbol{\Omega}_\perp]|^2 = \prod_k |\,\mathrm{Det}[\boldsymbol{\omega}_k]|^2 \neq 0, \tag{3.3}$$

i.e., $\boldsymbol{\Omega} = \mathbf{Z}^{\mathrm{T}}\mathbf{II}\mathbf{Z}$ is a nonsingular matrix. Eq. (3.2c) then implies that $\mathbf{Z}$ is also nonsingular. Hence the eight columns of $\mathbf{Z}$ are independent eigenvectors, and the eight associated eigensolutions combine to form the general solution of the piezoelectric material.

The inverse matrix of $\boldsymbol{\Omega}$ is also block diagonal, and may be expressed in terms of the inverses of $\boldsymbol{\omega}_k$ and $\bar{\boldsymbol{\omega}}_k$:

$$\boldsymbol{\Omega}^{-1} = <\boldsymbol{\Omega}_\perp^{-1},\bar{\boldsymbol{\Omega}}_\perp^{-1}> = << \boldsymbol{\omega}_k^{-1}>, <\bar{\boldsymbol{\omega}}_k^{-1}>>. \tag{3.4}$$

Eq. (3.2b) gives the explicit analytical expression of $\mathbf{Z}^{-1}$:

$$\mathbf{Z}^{-1} = \boldsymbol{\Omega}^{-1}\mathbf{Z}^{\mathrm{T}}\mathbf{II}. \tag{3.5}$$

$\mathbf{Z}^{-1}$ appears frequently in the expressions of Green's functions and singularity solutions including cracks and multi-material singularities.

For each $\mathbf{X}_k$ and $\overline{\mathbf{X}}_k$, consider the 8×8 symmetric matrices

$$\mathbf{X}_k\boldsymbol{\omega}_k^{-1}\mathbf{X}_k^{\mathrm{T}} = \mathbf{F}_k + \mathrm{i}\mathbf{G}_k, \quad \overline{\mathbf{X}}_k\overline{\boldsymbol{\omega}}_k^{-1}\overline{\mathbf{X}}_k^{\mathrm{T}} = \mathbf{F}_k - \mathrm{i}\mathbf{G}_k. \tag{3.6a,b}$$

$\mathbf{F}_k$ and $\mathbf{G}_k$ are the real and imaginary parts of $\mathbf{X}_k\boldsymbol{\omega}_k^{-1}\mathbf{X}_k^{\mathrm{T}}$ and hence they are also symmetric. Postmultiplying Eq. (3.6a) by $\mathbf{II}\mathbf{X}_k$ and using Eq. (2.1), one obtains

$$\mathbf{X}_k = (\mathbf{F}_k + \mathrm{i}\mathbf{G}_k)\mathbf{II}\mathbf{X}_k. \tag{3.7}$$

Therefore, the linear transformation $(\mathbf{F}_k + \mathrm{i}\mathbf{G}_k)\mathbf{II}$ maps every vector in $\mathbf{X}_k$ into itself. On the other hand, if $\boldsymbol{\xi}'$ is an eigenvector associated with a different eigenvalue, then it is orthogonal to all columns of $\mathbf{X}_k$, so that Eq. (3.6a) yields

$$(\mathbf{F}_k + \mathrm{i}\mathbf{G}_k)\mathbf{II}\boldsymbol{\xi}' = \mathbf{0}. \tag{3.8}$$

Therefore, $(\mathbf{F}_k + \mathrm{i}\mathbf{G}_k)\mathbf{II}$ is the projection operator into the eigenspace of μ_k, i.e., any eight-dimensional vector $\mathbf{v}$ may be decomposed as $\mathbf{v} = (\mathbf{F}_k + \mathrm{i}G_k)\mathbf{II}\mathbf{v} + \mathbf{v}^*$, where the first part belongs to the eigenspace and $\mathbf{v}*$ is orthogonal to it. Similarly, $(\mathbf{F}_k - \mathrm{i}\mathbf{G}_k)\mathbf{II}$ is the projection operator into the eigenspace of $\bar{\mu}_k$. This yields the decomposition of the identity matrix into orthogonal projections:

$$\mathbf{I}_8 = \sum_k (\mathbf{F}_k + \mathrm{i}\mathbf{G}_k)\mathbf{II} + \sum_k (\mathbf{F}_k - \mathrm{i}\mathbf{G}_k)\mathbf{II} = 2\left(\sum_k \mathbf{F}_k\right)\mathbf{II},$$

i.e.,

$$2\left(\sum_k \mathbf{F}_k\right) = \mathbf{II}, \tag{3.9}$$

where the summation extends over all eigenvalues with $\mathrm{Im}[\mu_k] > 0$.

Combining Eqs. (3.7) and (3.8), one has $(\mathbf{F}_k + \mathrm{i}\mathbf{G}_k)\mathbf{IIZ} = \mathbf{Z}_k$, where $\mathbf{Z}_k$ is the 8×8 matrix obtained from $\mathbf{Z}$ by setting all column vectors to $\mathbf{0}$ except those belonging to $\mathbf{X}_k$, which are left unchanged. Then

$$\mathbf{F}_k + \mathrm{i}\mathbf{G}_k = \mathbf{Z}_k\mathbf{Z}^{-1}\mathbf{II} = \mathbf{Z}_k\mathbf{\Omega}^{-1}\mathbf{Z}^{\mathrm{T}}. \tag{3.10}$$

These expressions allow direct determination of $\mathbf{F}_k$ and $\mathbf{G}_k$ in terms of the base matrix $\mathbf{Z}$.

Summing Eq. (3.6) over all eigenvalues with positive imaginary parts, one obtains

$$\mathbf{Z}_\perp\mathbf{\Omega}_\perp^{-1}\mathbf{Z}_\perp^{\mathrm{T}} = \sum_k \mathbf{F}_k + \mathrm{i}\sum_k \mathbf{G}_k. \tag{3.11}$$

Hence

$$2\mathrm{Re}\left[\mathbf{Z}_\perp\mathbf{\Omega}_\perp^{-1}\mathbf{Z}_\perp^{\mathrm{T}}\right] = 2\left(\sum_k \mathbf{F}_k\right) = \mathbf{II}, \tag{3.12a}$$

The various equalities relating the matrices $\mathbf{F}_k$ and $\mathbf{G}_k$ in the theory of coupled anisotropic plates (see Eqs. (64)–(66) in Yin, 2003a) are also formally valid in the present theory of piezoelectricity. The equalities imply the relation $\mathbf{\Gamma\,II\,\Gamma\,II} = -\mathbf{I}_8$ for the matrix $\mathbf{\Gamma}$, which is defined by Eq. (3.12b) and has special importance to the infinite space and half-space domains. Equivalently,

$$\mathbf{\Gamma}^{-1} = -\mathbf{II\Gamma II}. \tag{3.13}$$

From Eqs. (3.2), (3.11) and (3.12), one obtains

$$\mathbf{Z} < \mathbf{\Omega}_\perp^{-1}, \bar{\mathbf{\Omega}}_\perp^{-1} > \mathbf{Z}^{\mathrm{T}} = \mathbf{Z\Omega}^{-1}\mathbf{Z}^{\mathrm{T}} = \mathbf{II},\ \mathbf{Z} < \mathbf{\Omega}_\perp^{-1},\ -\bar{\mathbf{\Omega}}_\perp^{-1} > \mathbf{Z}^{\mathrm{T}} = \mathrm{i}\mathbf{\Gamma}. \tag{3.14a,b}$$

Postmultiplication of the last equation by $-\mathrm{i}\mathbf{II}$ yields

$$\mathbf{\Gamma II} = -\mathrm{i}\mathbf{Z} < \mathbf{\Omega}_\perp^{-1},\ -\bar{\mathbf{\Omega}}_\perp^{-1} > \mathbf{Z}^{\mathrm{T}}\mathbf{IIZZ}^{-1} = \mathbf{Z} < -\mathrm{i}\mathbf{I}_4, \mathrm{i}\mathbf{I}_4 > \mathbf{Z}^{-1}. \tag{3.15}$$

The last equation shows that $\mathbf{\Gamma}$ is completely determined by the matrix of eigenvectors. The equation is particularly useful in the following analysis for calculating $\mathbf{\Gamma}$ from $\mathbf{Z}$.

As in the two related theories of anisotropic elasticity and coupled anisotropic plates, the eigenvectors transform in the following manner

$$\mathbf{Z}^* = \mathbf{ZQ}_8, \quad \mathbf{Q}_8 = < \mathbf{Q}_2, \mathbf{I}_2, \mathbf{Q}_2, \mathbf{I}_2 > \tag{3.16a,b}$$

under a coordinate transformation from $\{x,y\}^{\mathrm{T}}$ to $\{x^*,y^*\}^{\mathrm{T}}$:

$$\begin{Bmatrix} x^* \\ y^* \end{Bmatrix} = \mathbf{Q}_2\begin{Bmatrix} x \\ y \end{Bmatrix}, \quad \text{where } \mathbf{Q}_2 \equiv \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{3.17}$$

The matrix $\mathbf{\Omega}$ remains unchanged under the coordinate transformation, whereas $\mathbf{\Gamma}$ and its submatrices transform according to tensorial rules:

$$\mathbf{\Gamma}^* = \mathbf{Q}_8\mathbf{\Gamma}\mathbf{Q}_8^{\mathrm{T}}, \quad \mathbf{L}^* = \mathbf{Q}_4\mathbf{L}\mathbf{Q}_4^{\mathrm{T}}, \quad \mathbf{H}^* = \mathbf{Q}_4\mathbf{H}\mathbf{Q}_4^{\mathrm{T}}, \quad \mathbf{S}^* = \mathbf{Q}_4\mathbf{S}\mathbf{Q}_4^{\mathrm{T}}, \quad \mathbf{Q}_4 \equiv < \mathbf{Q}_2, \mathbf{I}_2 >. \tag{3.18}$$

The matrices $\mathbf{G}_k$ and their 4×4 submatrices satisfy the same tensorial transformation rules.

Next, we consider an affine transformation in the eigenspace of an eigenvalue μ_k, characterized by a nonsingular complex matrix $\boldsymbol{\tau}_k$:

$$\mathbf{X}_k^* = \mathbf{X}_k\boldsymbol{\tau}_k. \tag{3.19}$$

The pseudometric transforms as follows

$$\boldsymbol{\omega}_k^* = [\![\mathbf{X}_k^*,\mathbf{X}_k^*]\!] = \boldsymbol{\tau}_k^{\mathrm{T}}[\![\mathbf{X}_k,\mathbf{X}_k]\!]\boldsymbol{\tau}_k = \boldsymbol{\tau}_k^{\mathrm{T}}\boldsymbol{\omega}_k\boldsymbol{\tau}_k, \tag{3.20}$$

and $\boldsymbol{\omega}_k^*$ has the inverse matrix satisfying

$$\mathbf{X}_k^*\boldsymbol{\omega}_k^{*-1}\mathbf{X}_k^{*\mathrm{T}} = \mathbf{X}_k\boldsymbol{\omega}_k^{-1}\mathbf{X}_k^{\mathrm{T}} = \mathbf{F}_k + \mathrm{i}\mathbf{G}_k. \tag{3.21}$$

If the affine transformation $\boldsymbol{\tau}_k$ in the eigenspace of μ_k is extended to an affine transformation $\boldsymbol{\tau}$ of the full eight-dimensional space such that $\boldsymbol{\tau}$ preserves every eigenspace, then Eqs. (3.20) and (3.21) are still valid after replacing $\boldsymbol{\tau}_k$ by $\boldsymbol{\tau}$. Hence we have the following theorem:

Theorem 3. *The projection operators* $(\mathbf{F}_k + \mathrm{i}\mathbf{G}_k)\mathbf{II}$ *are unchanged under any affine transformation that preserves all eigenspaces. The matrices* $\mathbf{F}_k$, $\mathbf{G}_k$, $\boldsymbol{\Gamma}$ *and their* 4×4 *submatrices satisfy the tensorial transformation rules of Eq. (3.18). These tensors are invariant under affine transformations that preserve the eigenspaces.*

The real, symmetric matrices $\mathbf{F}_k$, $\mathbf{G}_k$ and $\boldsymbol{\Gamma}$ are called intrinsic tensors because they are completely determined by the constitutive parameters. Unlike the pseudometrics and $\boldsymbol{\Omega}$, they are not dependent on the particular choice of the eigenvectors, i.e., they may be obtained from Eqs. (3.10) and (3.15) using any set of independent eigenvectors, and the results are always the same.

Theorems 1–3 are spectral theorems. They describe the spectral decomposition of the eight-dimensional complex vector space into orthogonal eigenspaces associated with distinct eigenvalues. The intrinsic geometrical structure of the space is determined by the pseudometrics, but the pseudometrics are not unique and cannot be made unique by normalization, because the complex space has no length or distance measure. However, the projection operators of Eqs. (3.6)–(3.8) determine the (Fourier) decomposition of every complex vector into its spectral components in the various eigenspaces. Eqs. (3.12a) show $(\mathbf{II} + \mathrm{i}\boldsymbol{\Gamma})/2$ as the spectral sum of $\mathbf{F}_k + \mathrm{i}\mathbf{G}_k$.

In Part I of this paper, the pseudometric $\boldsymbol{\omega}_k$ has been determined for eigenvalues of all types and multiplicities. $\boldsymbol{\omega}_k$ was shown to be nonsingular, and the explicit expression of the inverse matrix $\boldsymbol{\omega}_k^{-1}$ is easily obtained. Then Eq. (3.4) gives $\boldsymbol{\Omega}^{-1}$, and Eqs. (3.10) and (3.15) yield analytical expressions of $\mathbf{F}_k$, $\mathbf{G}_k$ and $\boldsymbol{\Gamma}$. The final expression of $\mathbf{G}_k$ may be given for each eigenspace in terms of the two functions $\delta(\mu)$ and $\mathbf{J}(\mu)\mathbf{W}(\mu)\mathbf{J}(\mu)^{\mathrm{T}}$ and their μ-derivatives of the various orders (all evaluated at $\mu = \mu_k$), as shown in the previous works on plane elasticity and coupled anisotropic plates (Yin, 2000a,b, 2003a). Such expressions have the merit of involving only material matrices, i.e., they do not involve quantities that vary with a different choice of the eigenvectors. From a practical viewpoint, however, Eq. (3.10) is more convenient and no less satisfactory. It too gives the *exact* and explicit expressions of $\mathbf{F}_k$ and $\mathbf{G}_k$ and, in addition, allows them to be easily computed from matrix products, without the need to evaluate the derivatives of $\delta(\mu)$ and $\mathbf{J}(\mu)\mathbf{W}(\mu)\mathbf{J}(\mu)^{\mathrm{T}}$.

In 2-D anisotropic elasticity, the base matrix $\mathbf{Z}$ and the intrinsic tensor $\boldsymbol{\Gamma}$ have the dimension 6×6, and it is customary to express $\mathbf{Z}$ and $\boldsymbol{\Gamma}$ in terms of 3×3 submatrices (the submatrices of $\boldsymbol{\Gamma}$ are often called Barnett–Lotte tensors though one of them already appeared in Stroh's paper, 1958). In the theory of coupled anisotropic plates (Yin, 2003a) and in the present theory of piezoelectricity, the symmetric matrices $\mathbf{G}_k$ and $\boldsymbol{\Gamma}$ may all be separated into 4×4 submatrices as follows:

$$\mathbf{\Gamma} \equiv \begin{bmatrix} -\mathbf{L} & \mathbf{S}^{\mathrm{T}} \\ \mathbf{S} & \mathbf{H} \end{bmatrix} \equiv 2\sum_k \mathbf{G}_k, \quad \mathbf{G}_k \equiv \begin{bmatrix} -\mathbf{L}_k & \mathbf{S}_k^{\mathrm{T}} \\ \mathbf{S}_k & \mathbf{H}_k \end{bmatrix} \tag{3.22a,b}$$

Then $\mathbf{L}_k$, $\mathbf{H}_k$, $\mathbf{L}$ and $\mathbf{H}$ are all real, symmetric matrices and Eq. (3.13) implies

$$\mathbf{HL} - \mathbf{SS} = \mathbf{LH} - \mathbf{S}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}} = \mathbf{I}_4, \quad \mathbf{LS} = -(\mathbf{LS})^{\mathrm{T}}, \quad \mathbf{SH} = -(\mathbf{SH})^{\mathrm{T}}. \tag{3.23a,b,c}$$

Hence **LS** and **SH** are skew-symmetric matrices while $\mathbf{S}_k$ and **S** are generally not symmetric. **L** and **H** are non-singular, and $\mathbf{L} < 1,1,1,-1>$ is positive definite (see Section 6). By deleting the fourth rows and the fourth columns of **L**, **H** and **S**, one obtains the well-known Barnett–Lothe tensors in 2-D anisotropic elasticity (Ting, 1996).

Let $\mathbf{X}_k$ and **Z** be expressed as follows in terms of submatrices of row dimension 4:

$$\mathbf{X}_k = \begin{bmatrix} \mathbf{B}_k \\ \mathbf{A}_k \end{bmatrix}, \quad \mathbf{Z} = \begin{bmatrix} \mathbf{B} & \overline{\mathbf{B}} \\ \mathbf{A} & \overline{\mathbf{A}} \end{bmatrix}. \tag{3.24}$$

Then Eqs. (3.6) and (3.22) yield

$$\mathbf{L}_k = \mathrm{Re}[\mathrm{i}\mathbf{B}_k\boldsymbol{\omega}_k^{-1}\mathbf{B}_k^{\mathrm{T}}], \quad \mathbf{H}_k = \mathrm{Re}[-\mathrm{i}\mathbf{A}_k\boldsymbol{\omega}_k^{-1}\mathbf{A}_k^{\mathrm{T}}], \quad \mathbf{S}_k = \mathrm{Re}[-\mathrm{i}\mathbf{A}_k\boldsymbol{\omega}_k^{-1}\mathbf{B}_k^{\mathrm{T}}], \tag{3.25}$$

whereas Eq. (3.16b) becomes

$$\mathbf{L} = 2\mathrm{i}\mathbf{B}\boldsymbol{\Omega}_{\perp}^{-1}\mathbf{B}^{\mathrm{T}}, \quad \mathbf{H} = -2\mathrm{i}\mathbf{A}\boldsymbol{\Omega}_{\perp}^{-1}\mathbf{A}^{\mathrm{T}}, \quad \mathbf{S} = -\mathrm{i}(2\mathbf{A}\boldsymbol{\Omega}_{\perp}^{-1}\mathbf{B}^{\mathrm{T}} - \mathbf{I}_4). \tag{3.26}$$

The relation $\mathbf{\Gamma\Pi Z}_{\perp} = -\mathrm{i}\mathbf{Z}_{\perp}$ yields $(\mathbf{S}^{\mathrm{T}} + \mathrm{i}\mathbf{I}_4)\mathbf{B} = \mathbf{LA}$ and $(\mathbf{S} + \mathrm{i}\mathbf{I}_4)\mathbf{A} = -\mathbf{HB}$. Hence if $\mathbf{\Gamma}$ is known, then **B** and **A** may be expressed in terms of each other:

$$\mathbf{A} = -(\mathbf{S} - \mathrm{i}\mathbf{I}_4)\mathbf{L}^{-1}\mathbf{B}, \quad \mathbf{B} = (\mathbf{S}^{\mathrm{T}} - \mathrm{i}\mathbf{I}_4)\mathbf{H}^{-1}\mathbf{A}, \tag{3.27}$$

so that the base matrix **Z** is completely determined by $\mathbf{\Gamma}$ and either **B** or **A**.

Materials with certain types of symmetry may possess equilibrium solutions in which some groups of eigensolutions do not contribute. For example, the in-plane and out-of-plane solutions of isotropic elastic materials are uncoupled. Each one of such solutions is unaffected by the intrinsic tensors of the nonparticipating eigenspaces, and its analytical results depend only on the eigenvectors and intrinsic tensors $\mathbf{G}_k$ of the remaining eigenspaces $\mathbf{X}_k$.

4. Material classification; general solution

4.1. Mathematically distinct types of eigenvalues

In Part I of the paper, a complete set of eight independent eigensolutions is obtained explicitly for every type of piezoelectric material. By combining the four complex conjugate pairs of eigensolutions, one obtains the general solution of two-dimensional equilibrium problems of that type of material. The general solution provides the foundation for theoretical analysis of piezoelectricity, in the same way that Goursat's representation of biharmonic functions in terms of a pair of complex analytic functions is fundamental to 2-D isotropic elasticity. It is also indispensable in most of analytical and numerical methods for solving boundary value problems, including power series, mapping of the domain boundary to a unit circle, analytical continuation, the integral equation approach and the boundary element method. The boundary element method requires, as a starting point, Green's function of the infinite domain. For elastic materials without piezoelectric effects, it has been found that material degeneracy significantly affects the singular part of Green's function, resulting in a changed angular variation of the singularity but not in the power of singularity (Yin, 2004).

In 2-D anisotropic elasticity, there are five distinct classes of materials, each requiring a different representation of the general solution (Yin, 2000a). These five classes of materials correspond to five possible combinations of five types of eigenvalues (normal eigenvalues with multiplicities 1, 2 and 3, and abnormal eigenvalues with multiplicities 2 and 3). In the theory of unsymmetric anisotropic plates, there are 11 distinct types of plates obtained by combining eigenvectors of eight different types: normal eigenvalues with multiplicities 1, 2, 3 and 4; abnormal eigenvalues with multiplicities 2, 3 and 4, and superabnormal eigenvalues (Yin, 2003b). In the present theory of two-dimensional piezoelectricity, the number of mathematically distinct types of eigenvalues is 11, as listed in Appendix B of Part I.

If two eigenvalues belong to the same one of the 11 types, then their eigenspaces have the same mathematical structure, and the eigensolutions have the same analytical forms of expression. However, each distinct type of eigenvalues in the mathematical sense may include several physically different cases. For example, for an eigenvector $\boldsymbol{\xi} = \mathbf{J}(\mu_0)\mathbf{W}(\mu_0)\boldsymbol{\rho}$ associated a simple eigenvalue $\mu_{0,}$ the meaning depends on the particular nonvanishing column $\mathbf{W}(\mu_0)\boldsymbol{\rho}$ of the matrix $\mathbf{W}(\mu_0)$ (where $\boldsymbol{\rho}$ denotes the corresponding column of $\mathbf{I}_3$). For a double abnormal eigenvalue μ_0 the two eigenvectors have the general expressions $\mathbf{J}(\mu_0)\boldsymbol{\eta}_1$ and $\mathbf{J}(\mu_0)\boldsymbol{\eta}_2$, where $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ are given by Eqs. (I-3.10a,b,c), respectively, if the first, second and the third diagonal element of $\mathbf{M}(\mu_0)$ does not vanish. These three cases yield, respectively, eigensolutions that show (a) no piezoelectric effect in antiplane deformation; (b) no piezoelectric effect in in-plane deformation and (c) no mechanical coupling between in-plane and anti-plane deformation. Thus, eigenvalues of the same mathematical type may correspond to different physical cases.

4.2. Classification of piezoelectric materials as two-dimensional media

The 2-D general solutions of various types of piezoelectric materials are determined by all possible combinations of the 11 distinct types of eigenvalues with a combined multiplicity equal to 4 (counting only eigenvalues with positive imaginary parts). An eigenvalue with multiplicity $p = 3$ can only combine with a simple eigenvalue, while one with $p = 4$ needs no other. Hence there are as many different types of eigenvalues with $p \geqslant 3$ as there are classes of materials that possess such eigenvalues. A double abnormal eigenvalue may combine with two simple eigenvalues, or with one double normal eigenvalue, or with another double abnormal eigenvalue. Finally, there are five classes of materials whose eigenvalues are all normal (one quadruple, one triple plus one simple, one double plus two simple, two double, and four simple). Hence the total number of distinct classes of piezoelectric materials is 14.

These 14 classes are shown below, characterized by the respective sets of eigenvalues. The symbol 1≅ stands for a simple eigenvalue; Np≅ for a normal one of multiplicity p; A2≅ for a double abnormal eigenvalue; A$p\alpha$,≅ for an abnormal-α eigenvalue of multiplicity p ($p = 3$ or 4) and A$p\beta$≅ for an abnormal-β eigenvalue. Finally, $S3''$ and $S4''$ denote, respectively, superabnormal eigenvalues of multiplicities 3 and 4. The symbols for different eigenvalues are separated by dashes. The degree of degeneracy is 4 minus the number of zeroth-order eigensolutions in $\mathbf{Z}_{\perp}$.

Nondegenerate materials: (1) 1-1-1-1; (2) 1-1-A2; (3) A2-A2; (4) 1-S3.
Degenerate of degree 1: (5) 1-1-N2; (6) N2-A2; (7) 1-A3α; (8) 1-A3β; (9) S4.
Degenerate of degree 2: (10) 1-N3; (11) N2-N2; (12) A4α; (13) A4β.
Degenerate of degree 3: (14) N4.

4.3. Kernel matrices and general solutions of 14 classes of piezoelectric materials

For the 11 distinct types of eigenvalues, we define the respective differential operators

$$\mathbf{D}_1 \equiv \mathbf{I}_1, \quad \mathbf{D}_{N2} \equiv \begin{bmatrix} 1 & d/d\mu \\ 0 & 1 \end{bmatrix}, \quad \mathbf{D}_{N3} \equiv \begin{bmatrix} 1 & d/d\mu & d^2/d\mu^2 \\ 0 & 1 & 2d/d\mu \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{D}_{N4} \equiv \begin{bmatrix} 1 & d/d\mu & d^2/d\mu^2 & d^3/d\mu^3 \\ 0 & 1 & 2d/d\mu & 3d^2/d\mu^2 \\ 0 & 0 & 1 & 3d/d\mu \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{D}_{A2} \equiv \mathbf{I}_2, \quad \mathbf{D}_{A3\alpha} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2d/d\mu \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{D}_{A4\beta} \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2d/d\mu & 3d^2/d\mu^2 \\ 0 & 0 & 1 & 3d/d\mu \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{D}_{A3\alpha} \equiv \begin{bmatrix} 1 & d/d\mu & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{D}_{A4\beta} \equiv \begin{bmatrix} 1 & d/d\mu & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3d/d\mu \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{D}_{S3} \equiv \mathbf{I}_3, \quad \mathbf{D}_{S4} \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3d/d\mu \\ 0 & 1 & 0 & 1 \end{bmatrix} \tag{4.1}$$

Then all equilibrium solutions that belong to the eigenspace $\mathbf{X}_k$ of each one of the 11 types of eigenvalues may be expressed by

$$\boldsymbol{\chi} = \mathbf{X}_k \|f\|_k \mathbf{c}_{k\perp}, \tag{4.2a}$$

where $\mathbf{c}_{k\perp}$ is an arbitrary complex constant vector of dimension p, and $\|f\|_k$ denotes the following expression with the right-hand side evaluated at the eigenvalue μ_k:

$$\|f\|_k = \mathbf{D}_k < f_1(x+\mu y,\mu),..,f_p(x+\mu y,\mu) >. \tag{4.2b}$$

Here the differential operator $\mathbf{D}_k$ is the one in Eq. (4.1) appropriate to μ_k, and the symbol $<f_1\ldots,f_p>$ denotes a diagonal matrix with p arbitrary analytic functions as the diagonal elements.

For each one of the 14 classes of piezoelectric materials, the 2-D general solution is obtained by combining the solutions of Eq. (4.2) of all eigenspaces belonging to $\mathbf{Z}_\perp$. Let the constant vectors $\mathbf{c}_{k\perp}$ be joined together to form a four-dimensional vector $\mathbf{c}_\perp$, and let the latter be joined with its complex conjugate vector to form the eight-dimensional vector $\mathbf{c}$:

$$\mathbf{c} \equiv \{\mathbf{c}_\perp^{\mathrm{T}},\bar{\mathbf{c}}_\perp^{\mathrm{T}}\}^{\mathrm{T}}. \tag{4.3}$$

Furthermore, let the matrices $\|f\|_k$ be joined to form a 4×4 block-diagonal matrix $\|f\|_\perp$, and define the 8×8 kernel matrix

$$\|f\| \equiv < \|f\|_\perp,\# >. \tag{4.4}$$

Then the general solution of 2-D piezoelectricity is given by $\boldsymbol{\chi} = 2\mathrm{Re}[\mathbf{Z}_\perp\|f\|_\perp\mathbf{c}_\perp]$ or, equivalently,

$$\boldsymbol{\chi} = \mathbf{Z}\|f(x+\mu y,\mu)\|\mathbf{c}. \tag{4.5}$$

In Eq. (4.4) and in the following, $<\boldsymbol{\Phi},\#>$ denotes the block-diagonal matrix having $\boldsymbol{\Phi}$ and its complex conjugate matrix as the two diagonal blocks.

The elements of the constant vector $\mathbf{c}$ may be absorbed into the arbitrary analytic functions. However, in many applications, all analytic functions in the kernel matrix differ only by constant multiplication factors. Then it is convenient to use one single function plus the vector $\mathbf{c}$. Eqs. (3.2a) and (4.3) imply that

$\mathbf{h} \equiv \mathbf{Zc} = \mathbf{Z}_\perp \mathbf{c}_\perp + \overline{\mathbf{Z}}_\perp \bar{\mathbf{c}}_\perp$ is real. Hence Eq. (4.5) may be recast in the following form in terms of a *real* constant vector $\mathbf{h}$:

$$\boldsymbol{\chi} = \mathbf{Z}\|f(z,\mu)\|\mathbf{Z}^{-1}\mathbf{h}, \quad z \equiv x + \mu y, \tag{4.6}$$

Indeed it is easily verified, using (3.2a), (3.4), (3.5) and (4.4), that $\mathbf{Z}\|f(z,\mu)\|\mathbf{Z}^{-1}$ is real for arbitrary complex functions f.

For any two functions f and g, Eqs. (4.1) and (4.2) imply that

$$\|fg\|_k = \|f\|_k \|g\|_k. \tag{4.7}$$

The proof follows from using the various differential operators in Eq. (4.1), and is not trivial unless μ_k is a simple eigenvalue. From Eq. (4.7) it is easily seen that $\|f\,g\|_\perp = \|f\|_\perp \|g\|_\perp$. Hence

$$\|fg\| = \|f\|\|g\|. \tag{4.8}$$

Taking the gradient of Eq. (4.5), one obtains

$$\boldsymbol{\chi}_{,x} = \{-\tau_{xy}, -\sigma_y, -\tau_{yz}, -D_y, \varepsilon_x, v_{,x}, w_{,x}, -E_x\}^{\mathrm{T}} = \mathbf{Z}\|f_{,z}\|\mathbf{Z}^{-1}\mathbf{h}, \tag{4.9a}$$

$$\boldsymbol{\chi}_{,y} = \{\sigma_x, \tau_{xy}, \tau_{xz}, D_x, u_{,y}, \varepsilon_y, w_{,y}, -E_y\}^{\mathrm{T}} = \mathbf{Z}\|\mu f_{,z}\|\mathbf{Z}^{-1}\mathbf{h} = \mathbf{Z}\|\mu\|\|f_{,z}\|\mathbf{Z}^{-1}\mathbf{h}. \tag{4.9b}$$

Define the diagonal matrix

$$\boldsymbol{\Delta} \equiv \langle 1,1,1,-1\rangle. \tag{4.10}$$

Eq. (4.9a) becomes

$$\{\varepsilon_x, v_{,x}, w_{,x}, E_x, \tau_{xy}, \sigma_y, \tau_{yz}, -D_y\}^{\mathrm{T}} = \langle \boldsymbol{\Delta}, -\boldsymbol{\Delta}\rangle \mathbf{IIZ}\|f_{,z}\|\mathbf{Z}^{-1}\mathbf{h}.$$

Taking the scalar product with (4.9b), one obtains twice the energy density function

$$2U_0 = \sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \tau_{xy}\gamma_{xy} + \tau_{xz}w_{,x} + \tau_{yz}w_{,y} + D_xE_x + D_yE_y = \mathbf{f}^{\mathrm{T}}\mathscr{H}\mathbf{f}, \tag{4.11a}$$

$$\mathscr{H} \equiv \langle \boldsymbol{\Delta}, -\boldsymbol{\Delta}\rangle \mathbf{Z}\|\mu\|\mathbf{Z}^{-1}\mathbf{II} = \langle \boldsymbol{\Delta}, -\boldsymbol{\Delta}\rangle \mathbf{Z}\|\mu\|\boldsymbol{\Omega}^{-1}\mathbf{Z}^{\mathrm{T}}, \mathbf{f} \equiv \mathbf{IIZ}\|f_{,z}\|\mathbf{Z}^{-1}\mathbf{h}. \tag{4.11b,c}$$

Notice that the real vector function $\mathbf{IIf}$ differs from the complex function $\boldsymbol{\chi}$ only in the replacement of f by $f_{,z}$. The real matrix $\mathscr{H}$ depends on the material parameters only, and it is required to be positive definite. It may be expressed as the following spectral sum:

$$\mathscr{H} = \langle \boldsymbol{\Delta}, -\boldsymbol{\Delta}\rangle \sum_k (\mathbf{X}_k \|\mu\|_k \boldsymbol{\omega}_k^{-1} \mathbf{X}_k^{\mathrm{T}} + \overline{\mathbf{X}}_k \|\bar{\mu}\|_k \bar{\boldsymbol{\omega}}_k^{-1} \overline{\mathbf{X}}_k^{\mathrm{T}}). \tag{4.12}$$

Since only the symmetric part of $\mathscr{H}$ affects the energy function, one may replace $\mathscr{H}$ in Eq. (4.11a) by $(\mathscr{H} + \mathscr{H}^{\mathrm{T}})/2$.

If the material is nondegenerate (including the case of four distinct eigenvalues μ_1, μ_2, μ_3 and μ_4), then the kernel matrix $\|f\|$ is diagonal, and $\mathbf{Z}$ contains no higher-order eigenvectors. Then the general solution reduces to the simple sum

$$\boldsymbol{\chi} = 2\mathrm{Re}\left[\sum_{1\leqslant k\leqslant 4} f_k(x + \mu_k y, \mu_k)\boldsymbol{\xi}_k\right], \tag{4.13}$$

and Eq. (4.12) becomes

$$\mathscr{H} = \langle \mathbf{\Delta}, -\mathbf{\Delta}\rangle \sum_k (\mu_k \mathbf{X}_k \boldsymbol{\omega}_k^{-1} \mathbf{X}_k^{\mathrm{T}} + \bar{\mu}_k \overline{\mathbf{X}}_k \bar{\boldsymbol{\omega}}_k^{-1} \overline{\mathbf{X}}_k^{\mathrm{T}}) = \langle \mathbf{\Delta}, -\mathbf{\Delta}\rangle \sum_k \{\mu_k(\mathbf{F}_k + \mathrm{i}\mathbf{G}_k) + \bar{\mu}_k(\mathbf{F}_k - \mathrm{i}\mathbf{G}_k)\}$$

$$= 2\langle \mathbf{\Delta}, -\mathbf{\Delta}\rangle \sum_k \{\mathrm{Re}[\mu_k]\mathbf{F}_k - \mathrm{Im}[\mu_k]\mathbf{G}_k\}. \quad (4.14)$$

5. Two-dimensional Green's function of the infinite space

As an application of the general solution of Eq. (4.4), we consider the 2-D equilibrium solution $\boldsymbol{\chi}$ of an infinite piezoelectric medium that satisfies the requirements:

(i) the stress field and electric field vanish at infinity;
(ii) $\boldsymbol{\chi}$ has a constant discontinuity $\boldsymbol{\chi}_0$ when crossing the negative x-axis, i.e., in polar coordinates,

$$[\boldsymbol{\chi}] \equiv \oint d\boldsymbol{\chi} = \boldsymbol{\chi}(r,\pi) - \boldsymbol{\pi}(r,-\pi) = \boldsymbol{\chi}_0, \quad (5.1)$$

where the integral is along a closed path encircling the origin in the counterclockwise sense. We let

$$\boldsymbol{\chi} = (2\pi)^{-1}\mathbf{G}_\infty \boldsymbol{\chi}_0. \quad (5.2)$$

Then the matrix function $(2\pi)^{-1}$ $\mathbf{G}_\infty$ is called Green's function of the infinite space with a line singularity along the z-axis through the origin, and the constant vector $\boldsymbol{\chi}_0$ is called the strength of singularity. These discontinuity conditions imply that the piezoelectric medium is subjected at $(x, y) = (0, 0)$ to concentrated line forces in three coordinate directions, a line charge, line dislocation of the displacements, and line discontinuity of the electric potential. By setting all analytic function in the kernel matrix to

$$f(z,\mu) = -\mathrm{i}\log[z], \quad z \equiv x + \mu y, \quad (5.3)$$

Eq. (4.5) yields

$$\boldsymbol{\chi} = \mathbf{Z}\| -\mathrm{i}\log[z]\|\mathbf{c}. \quad (5.4)$$

On the positive x-axis, the kernel matrix $\|-\mathrm{i}\log[z]\|$ reduces to a diagonal matrix, irrespective of the type of material, since μ-differentiation of $\log[z]$ generates the factor y. On the upper and lower sides of the negative x-axis, the kernel matrix is diagonal and has the values $\langle -\mathrm{i}(\log[r] + \mathrm{i}\pi)\mathbf{I}_3, \#\rangle$and $\langle \mathrm{i}(\log[r] - \mathrm{i}\pi)\mathbf{I}_3, \#\rangle$, respectively, since μ has a positive imaginary part. Consequently, as one crosses the negative x-axis from the lower side to the upper side, $\|-\mathrm{i}\log[x + \mu y]\|$ has a constant jump equal to $2\pi\mathbf{I}_8$. Hence Eq. (5.4) has a discontinuity equal to 2π $\mathbf{Zc}$. Let $\mathbf{c} = (2\pi)^{-1}\mathbf{Z}^{-1}\boldsymbol{\chi}_0$. Then Eqs. (5.2) and (5.4) yield *two-dimensional Green's function of the infinite space*

$$(1/2\pi)\mathbf{G}_\infty \equiv (1/2\pi)\mathbf{Z}\| -\mathrm{i}\log[x + \mu y]\|\mathbf{Z}^{-1}. \quad (5.5)$$

This expression is valid for all piezoelectric materials regardless of degeneracy. For degenerate materials, $\mathbf{Z}$ contains higher order eigenvectors, and the kernel matrix contains off-diagonal elements resulting from the application of the differential operators as defined by Eq. (4.1) for the various types of eigenvalues. Green's functions associated with all types of line discontinuities, including the line forces, dislocation of displacements, and discontinuities of the electric field, are included in a unified manner in one single expression.

Eq. (4.4) for the general solution and Eq. (5.5) for Green's function of the infinite space are formally identical to the corresponding expressions in anisotropic elasticity (Yin, 2004) and coupled anisotropic plate theory (Yin, in press a). However, the 8×4 matrix $\mathbf{Z}_\perp$ in the present theory may contain zeroth-order eigenvectors that require different expressions such as described in Theorems 7–10 of Part I, and the kernel matrix may involve new differential operators as shown in Eq. (4.1).

Using Eq. (3.15), one may rewrite Eq. (5.5) as

$$\mathbf{G}_\infty = \log[r]\mathbf{\Gamma II} + \mathbf{\Gamma IIZ}\| \log[\cos\theta + \mu\sin\theta]\|\mathbf{Z}^{-1}. \tag{5.6}$$

Hence Green's function of the infinite space is the sum of two parts, one depending only on r and the other only on θ. Notice that it is easy to invert the block-diagonal matrix $\mathbf{\Omega}$, as shown in Eq. (3.4). Then $\mathbf{Z}^{-1}$ in Eqs. (5.5) and (5.6) is easily obtained by matrix multiplication: $\mathbf{Z}^{-1} = \mathbf{\Omega}^{-1}\mathbf{Z}^{\mathrm{T}}\,\mathbf{II}$, and direct inversion of the 8×8 matrix $\mathbf{Z}$ is avoided.

Taking the gradient of Eq. (5.5), and making suitable combinations, one obtains the following solution associated with the strength of singularity $\boldsymbol{\chi}_0$

$$\{-\tau_{r\theta}, -\sigma_\theta, -\tau_{\theta z}, -D_\theta, \varepsilon_r, u_{\theta,r}, \gamma_{rz}, -E_r\}^{\mathrm{T}} = (1/2\pi r)\mathbf{Q}_8\mathbf{GII}\boldsymbol{\chi}_0, \tag{5.7}$$

where $\mathbf{Q}_8$ was defined by Eq. (3.16b). Hence all components of Eq. (5.7) have the same simple pattern and vary sinusoidal in θ. Other components of the stress, strain and electric fields generally have complicated θ-dependence due to the complex argument $\cos\theta + \mu_k\sin\theta$.

6. A half space with concentrated line loads or dislocations at a boundary point

From Green's function $\mathbf{G}_\infty$ of the last section, one may easily obtained a derived result for a half space $y \geqslant 0$, with traction-free boundary condition on $y = 0$ except for concentrated line loads or dislocation at a boundary point which is taken to be the origin of the (r,θ) coordinates.

Using Eqs. (5.6), (3.13) and (3.15), one obtains the boundary values of $\mathbf{G}_\infty$:

$$\mathbf{G}_\infty = \begin{cases} \log[r]\mathbf{GII} & \text{on } \theta = 0 \\ \log[r]\mathbf{GII} + \pi\mathbf{I}_8 & \text{on } \theta = \pi \end{cases} \tag{6.1}$$

Let

$$\boldsymbol{\chi} = (1/2\pi)\mathbf{G}_\infty\boldsymbol{\chi}_0 + \{P_1, P_2, P_3, \varsigma_0, 0, 0, 0, 0\}^{\mathrm{T}}, \quad \boldsymbol{\chi}_0 \equiv \begin{Bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{Bmatrix}, \tag{6.2}$$

where $\mathbf{f}_1$ and $\mathbf{f}_2$ are four-dimensional constant vectors to be determined from the requirement that the first four components of $\boldsymbol{\chi}$ vanish on $\theta = 0$, and assume the constant values P_1, P_2, P_3 and ς_0 respectively, on $\theta = \pi$. Substituting Eq. (6.2) into (6.1) and using (3.22a), one obtains

$$\mathbf{S}^{\mathrm{T}}\mathbf{f}_1 - \mathbf{L}\mathbf{f}_2 = \mathbf{0}, \quad 1/2\mathbf{f}_1 + \{P_1, P_2, P_3, \varsigma_0\}^{\mathrm{T}} = \mathbf{0}.$$

Hence

$$\boldsymbol{\chi}_0 \equiv -2\begin{bmatrix} \mathbf{I}_4 \\ \mathbf{L}^{-1}\mathbf{S}^{\mathrm{T}} \end{bmatrix}\{P_1, P_2, P_3, \varsigma_0\}^{\mathrm{T}}. \tag{6.3}$$

Eqs. (6.2) and (6.3) give the solution of the half space $y \geqslant 0$ with free boundary subjected to concentrated forces and charge at the origin. The physical variables are given by

$$\{-\tau_{xy}, -\sigma_y, -\tau_{yz}, -D_y, \varepsilon_x, v_{,x}, w_{,x}, -E_x\}^{\mathrm{T}} = (1/2\pi)\mathbf{Z}\| - \mathrm{i}/z\|\mathbf{Z}^{-1}\boldsymbol{\chi}_0, \tag{6.4a}$$

$$\{\sigma_x, \tau_{xy}, \tau_{xz}, D_x, u_{,y}, \varepsilon_y, w_{,y}, -E_y\}^{\mathrm{T}} = (1/2\pi)\mathbf{Z}\| - \mathrm{i}\mu/z\|\mathbf{Z}^{-1}\boldsymbol{\chi}_0. \tag{6.4b}$$

Furthermore, Eqs. (3.15), (3.23), (5.7) and (6.3) yield simple results in polar coordinates:

$$\tau_{r\theta} = \sigma_\theta = \tau_{\theta z} = D_\theta = 0, \tag{6.5a}$$

$$\{\varepsilon_r, u_{\theta,r}, \gamma_{rz}, -E_r\}^{\mathrm{T}} = -(1/2\pi r)\mathbf{Q}_4\mathbf{L}^{-1}\{P_1, P_2, P_3, \varsigma_0\}^{\mathrm{T}}. \tag{6.5b}$$

Thus, the simple radial distribution of stress, well-known in isotropic elasticity, is also valid in plane anisotropic elasticity regardless of material degeneracy.

On the positive x-axis, the four nonvanishing components of $\boldsymbol{\chi}$ are

$$\{u, v, w, \phi\}^{\mathrm{T}} = -(1/\pi)\log[r]\mathbf{L}^{-1}\{P_1, P_2, P_3, \varsigma_0\}^{\mathrm{T}}. \tag{6.6a}$$

On the negative x-axis, one has

$$\{u, v, w, \phi\}^{\mathrm{T}} = -(1/\pi)\{\mathbf{L}^{-1}\mathbf{S}^{\mathrm{T}} + \log[r]\mathbf{L}^{-1}\}\{P_1, P_2, P_3, \varsigma_0\}^{\mathrm{T}}. \tag{6.6b}$$

Consider the special class of materials whose symmetry properties imply that the antiplane deformation is completely uncoupled from the in-plane deformation and the electric effects, i.e., all coefficients of the polynomials $M_{12}(\mu)$ and $M_{23}(\mu)$ vanish [see Eq. (I-2.20)]. Under the additional assumption of non-degenerate materials, the solution of the half-space problem under a line load and a line charge at a boundary point was given by Sosa and Castro (1994) using the Fourier transform method. Due to their use of the stiffness-based formalism, the resulting expressions are significantly more complicated than our Eqs. (6.2)–(6.5), and contain 16 real constants (Φ_k, Ω_k and Ψ_k, in their notation) whose expressions in terms of basic constitutive constants are too long to be shown in their paper.

The numerical solutions of Sosa and Castro were given for three materials including Lead Zirconate Titanate (PZT-4). In an earlier paper, Sosa (1991) showed the material constants of PZT-4, and derived the analytical expression of the characteristic equation for the special class of materials with uncoupled antiplane behavior and with $\beta_{16} = \beta_{26} = \gamma_{11} = \gamma_{12} = \gamma_{26} = 0$. The general characteristic equation in the Appendix A of this paper reduces exactly to Sosa's equation for this special class of material. The present eigenvalues for PZT-4, ± 1.2185i and $\pm 0.2006 \pm 1.0699$i, are slightly different from those of Sosa and Castro (1994), *viz*., ± 1.204i and $\pm 0.2004 \pm 1.069$i. Sosa and Castro obtained $\sigma_y = -0.551P_2/r$ along the y-axis of the half space subjected to a vertical force P_2 at the origin, which is close to our solution $\sigma_y = -0.5480P_2/r$, and the magnitude of difference is comparable with those occurring in the eigenvalues.

The present solution involves only algebraic analysis, and is obtained straightforwardly from the infinite space solution. Furthermore, it is valid for all classes of nondegenerate and degenerate piezoelectric materials and for more general types of loads including dislocations. Eqs. (5.6) and (6.5) show, in an exceedingly lucid manner, the dependence of the solution upon the basic matrix $\mathbf{Z}$, the intrinsic tensor $\boldsymbol{\Gamma}$, and the material eigenvalues, as well as the simple dependence of $\tau_{r\theta}$, σ_θ, $\tau_{\theta z}$, D_θ, ε_r, $u_{\theta,r}$, γ_{rz} and $-E_r$ upon the angular coordinate θ.

If the origin is a center of dislocation rather than a center of force, and if the boundary conditions $u = v = w = \phi = 0$ are prescribed on the negative x-axis whereas their values are u_0, v_0, w_0 and ς_0 on the positive x-axis, then Eq. (6.3) is replaced by

$$\boldsymbol{\chi}_0 \equiv -2\begin{bmatrix}\mathbf{L}^{-1}\mathbf{S}^{\mathrm{T}} \\ \mathbf{I}_4\end{bmatrix}\{u_0, v_0, w_0, \phi_0\}^{\mathrm{T}}, \tag{6.7}$$

and the solutions of Eqs. (6.5a,b) are replaced by

$$\varepsilon_r = u_{\theta,r} = \gamma_{rz}, = E_r = 0, \tag{6.8a}$$

$$\{\tau_{r\theta}, \sigma_\theta, \tau_{\theta z}, D_\theta\}^{\mathrm{T}} = -(1/2\pi r)\mathbf{Q}_4\mathbf{H}^{-1}\{u_0, v_0, w_0, \varsigma_0\}^{\mathrm{T}}. \tag{6.8b}$$

From Eq. (6.6a) one obtains

$$P_1u + P_2v + P_3w - \varsigma_0\phi = -(1/\pi)\log[r]\{P_1,P_2,P_3,\varsigma_0\}\langle 1,1,1,-1\rangle \mathbf{L}^{-1}\{P_1,P_2,P_3,\varsigma_0\}^{\mathrm{T}}, \tag{6.9}$$

where the concentrated loads are applied at the origin but u, v, w and ϕ may be evaluated at any point in the half space. As (x,y) moves close to the origin, $-\log[r]$ becomes positive and $P_1u + P_2v + P_3w - \varsigma_0\phi$ must be positive since a negative ς_0 implies a positive $D_y = -\varsigma_{,x}$ accompanied by a negligibly small D_x in the vicinity, which in turn imply $E_y \geqslant 0$. Hence the matrix $\langle 1,1,1,-1\rangle \mathbf{L}^{-1}$ is positive definite.

7. Interface cracks

Consider two half spaces of different piezoelectric materials, perfectly bonded along the positive x-axis, and separated by a traction-free and charge-free crack along the negative x-axis. The material of the lower half space has the base matrix $\mathbf{Z}$ and eigenvalues $\{\mu\}$. The corresponding entities of the upper region material are $\mathbf{Z}'$ and $\{\mu'\}$. Similar to Eqs. (3.15) and (3.22a), one defines the intrinsic tensor $\mathbf{\Gamma}' = \mathbf{Z}'\langle -\mathrm{i}\mathbf{I}_4, \mathrm{i}\mathbf{I}_4\rangle Z'^{-1}$ $\mathbf{II}$ and its 4×4 submatrices $\mathbf{L}'$, $\mathbf{H}'$ and $\mathbf{S}'$. Consider the solutions in the lower and upper half spaces given respectively by

$$\boldsymbol{\chi} = \mathbf{Z}\|(x+\mu y)^{\lambda}\|\mathbf{Z}^{-1}\mathbf{f}, \quad \boldsymbol{\chi}' = \mathbf{Z}'\|(x+\mu' y)^{\lambda}\|\mathbf{Z}'^{-1}\mathbf{f}', \tag{7.1a,b}$$

where the undetermined constant λ is generally complex and is called an eigenvalue of the interface crack. Continuity condition on the positive x-axis implies $\mathbf{f}' = \mathbf{f}$, since the two kernel matrices of Eqs. (7.1a,b) both reduce to the identity matrix. Then, on the upper and lower crack faces, one has

$$\boldsymbol{\chi}'|_{\theta=\pi} = \mathbf{Z}'\langle \exp[\mathrm{i}\pi\lambda], \exp[-\mathrm{i}\pi\lambda]\rangle \mathbf{Z}'^{-1}\mathbf{f} = \sin\pi\lambda(ctn\pi\lambda\mathbf{I}_8 - \mathbf{\Gamma}'\mathbf{II})\mathbf{f}, \tag{7.2a}$$

$$\boldsymbol{\chi}|_{\theta=-\pi} = \mathbf{Z}\langle \exp[-\mathrm{i}\pi\lambda], \exp[\mathrm{i}\pi\lambda]\rangle \mathbf{Z}^{-1}\mathbf{f} = \sin\pi\lambda(ctn\pi\lambda\mathbf{I}_8 + \mathbf{\Gamma}\mathbf{II})\mathbf{f}, \tag{7.2b}$$

where Eq. (3.15) has been used. If the two crack faces are traction-free and charge-free, then

$$\sin\pi\lambda[\mathbf{I}_4, \mathbf{0}_{4\times 4}](ctn\pi\lambda\mathbf{I}_8 - \mathbf{\Gamma}'\mathbf{II})\mathbf{f} = \mathbf{0}, \tag{7.3a}$$

$$\sin\pi\lambda[\mathbf{I}_4, \mathbf{0}_{4\times 4}](ctn\pi\lambda\mathbf{I}_8 + \mathbf{\Gamma}\mathbf{II})\mathbf{f} = \mathbf{0}. \tag{7.3b}$$

These equations are satisfied when λ is an integer (i.e., $sin\pi\lambda = 0$), but then Eqs. (7.1a,b) yield nonsingular solutions that are polynomial functions of the coordinates. A different set of solutions of Eqs. (7.3a,b) is determined by

$$\mathrm{Det}\begin{bmatrix} ctn\pi\lambda\mathbf{I}_4 - \mathbf{S}'^{\mathrm{T}} & \mathbf{L}' \\ ctn\pi\lambda\mathbf{I}_4 + \mathbf{S}^{\mathrm{T}} & -\mathbf{L} \end{bmatrix} = 0,$$

or, equivalently,

$$\mathrm{Det}[ctn\pi\lambda\mathbf{I}_4 - (\mathbf{L}^{-1} + \mathbf{L}'^{-1})^{-1}(\mathbf{S}\mathbf{L}^{-1} - \mathbf{S}'\mathbf{L}'^{-1})] = 0, \tag{7.4}$$

i.e., $ctn\pi\lambda$ is an eigenvalue of the matrix $\mathbf{V} \equiv (\mathbf{L}^{-1} + \mathbf{L}'^{-1})^{-1}$ $(\mathbf{S}\mathbf{L}^{-1} - \mathbf{S}'\mathbf{L}'^{-1})$. Eq. (7.4) is formally identical to the characteristic equation of the bimaterial interface crack in 2-D anisotropic elasticity. However, $\mathbf{S}$, $\mathbf{L}$, $\mathbf{S}'$ and $\mathbf{L}'$ in the present problem have the dimension 4×4, instead of 3×3. Hence Eq. (7.4) possesses additional roots compared to the same equation for 2-D elasticity.

A 3×3 skew-symmetric matrix is always singular but a 4×4 skew-symmetric matrix is generally nonsingular. Let T_{ij} be the elements of the skew-symmetric matrix $\mathbf{T} \equiv \mathbf{S}\mathbf{L}^{-1} - \mathbf{S}'\mathbf{L}'^{-1}$. Then

$$\mathrm{Det}[\mathbf{T}] = \mathrm{Det}[\mathbf{S}\mathbf{L}^{-1} - \mathbf{S}'\mathbf{L}'^{-1}] = (T_{14}T_{23} + T_{12}T_{34} - T_{13}T_{24})^2 \geqslant 0. \tag{7.5}$$

As pointed out in the last section, $\langle 1,1,1,-1\rangle(\mathbf{L}^{-1}+\mathbf{L}'^{-1})$ is a positive definite matrix. Hence Det $[\mathbf{L}^{-1}+\mathbf{L}'^{-1}]<0$ and Det $[\mathbf{V}]=\text{Det}[\mathbf{U}]\text{Det}[\mathbf{T}]\leqslant 0$ where $\mathbf{U}\equiv(\mathbf{L}^{-1}+\mathbf{L}'^{-1})^{-1}$. If q is an eigenvalue of $\mathbf{V}$ with the eigenvector ς, then

$$(\mathbf{UT}-q\mathbf{I}_4)\varsigma=\mathbf{0}. \tag{7.6}$$

Since $\mathbf{U}$ is nonsingular, whether Eq. (7.6) may be satisfied by $q=0$ depends on whether Det $[\mathbf{T}]=0$, i.e., whether $T_{14}\,T_{23}+T_{12}T_{34}-T_{13}T_{24}=0$. Premultiplication of Eq. (7.6) by $\mathbf{T}$ yields $(\mathbf{TU}-q\mathbf{I}_4)\mathbf{T}\varsigma=\mathbf{0}$. This implies that $\mathbf{TU}-q\mathbf{I}_4$ and its transpose $-\mathbf{UT}-q\mathbf{I}_4$ are singular matrices. Therefore if $\mathbf{V}=\mathbf{UT}$ has the eigenvalue q, then it also has the eigenvalue $-q$. Hence one has the following characteristic equation associated with (7.6):

$$\text{Det}[\mathbf{V}-q\mathbf{I}_4]=q^4-2\kappa q^2-\sigma=0,\quad \sigma=-\text{Det}[\mathbf{V}]\geqslant 0 \tag{7.7}$$

where 2κ is the second invariant of $\mathbf{V}$. It follows that q^2 is real: $q^2=\kappa\pm(\kappa^2\pm\sigma^{1/2})$. Hence there is one pair of real roots and one pair of purely imaginary roots:

$$q_{1,2}=\pm\{\kappa+(\kappa^2+\sigma)^{1/2}\}^{1/2},\quad q_{3,4}=\pm\mathrm{i}\{-\kappa+(\kappa^2+\sigma)^{1/2}\}^{1/2}. \tag{7.8}$$

The results (7.7) and (7.8) were obtained by Suo et al. (1992) through a different derivation starting from the general solution of a nondegenerate material, and it is shown here that they remain valid in all degenerate cases.

If $\text{Det}[\mathbf{T}]=T_{14}T_{23}+T_{12}T_{34}-T_{13}T_{24}=0$, then $\sigma=0$ and Eq. (7.8) reduces to $q_{1,2}=\pm(2\kappa)^{1/2}$ and $q_{3,4}=0$. Then the eigenvalues of Eq. (7.4) are given by

$$\lambda=n-1/2\quad(\text{double eigenvalues}, n=1,2,\ldots), \tag{7.9a}$$

$$\lambda=n-1/2\pm(1/\pi)\cos^{-1}\{1/(1+2\kappa)^{1/2}\}\quad(n=1,2,\ldots). \tag{7.9b}$$

In the contrary case $\sigma>0$, the eigenvalues are

$$\lambda=n-1/2\pm(1/2\pi)\cos^{-1}[\{1-\kappa-(\kappa^2+\sigma)^{1/2}\}/\{1+\kappa+(\kappa^2+\sigma)^{1/2}\}], \tag{7.10a}$$

$$\lambda=n-1/2\pm(\mathrm{i}/2\pi)\log[(1+\{(\kappa^2+\sigma)^{1/2}-\kappa\}^{1/2})/(1-\{(\kappa^2+\sigma)^{1/2}-\kappa\}^{1/2})]. \tag{7.10b}$$

The parameters κ and σ are small compared to 1 if the piezoelectric effect is not severe. The inverse square root singularity and the near crack tip oscillatory behavior associated with complex eigenvalues of Eq. (7.10b) (with $n=1$) are well known. However, Eq. (7.10a) implies that the order of the dominant singularity is $\text{Re}[\lambda]<1/2$, which yields singularities of the stress and electric fields stronger than the inverse square root singularity.

If the crack faces are rigidly fixed ($u=v=w=0$) and grounded ($\phi=0$), then $[\mathbf{I}_4,\mathbf{0}_{4\times4}]$ in Eqs. (7.3a,b) must be replaced by $[\mathbf{0}_{4\times4},\mathbf{I}_4]$, so that the following characteristic equation replaces (7.4)

$$\text{Det}[ctn\pi\lambda\mathbf{I}_4-(\mathbf{H}^{-1}+\mathbf{H}'^{-1})^{-1}(\mathbf{H}'^{-1}\mathbf{S}'-\mathbf{H}^{-1}\mathbf{S})]=0. \tag{7.11}$$

Hence $ctn\pi\lambda$ is an eigenvalue of the matrix $(\mathbf{H}^{-1}+\mathbf{H}'^{-1})^{-1}(\mathbf{H}'^{-1}\mathbf{S}'-\mathbf{H}^{-1}\mathbf{S})$.

The characteristic Eqs. (7.4) and (7.11) are unaffected by material degeneracy. However, as usual, the elasticity solutions (7.1a,b) in the two half spaces depend on degeneracy and on the type of material in two essential ways: the base matrix $\mathbf{Z}$ of degenerate materials contains high-order eigenvectors, and the kernel matrix has off-diagonal elements implying the participation of lower-order eigenvectors in higher order eigensolutions.

Once the eigenvalues λ have been determined, Eqs. (7.1a,b) and (7.3a,b) give the corresponding eigensolutions of the interface crack. Boundary value problems involving the crack may be solved by combining the eigensolutions to fit the data on a circular path enclosing the crack tip (such data are obtained easily by numerical methods since the exterior region is not appreciably affected by the crack tip singularity), as shown in a recent work on the solution of anisotropic multimaterials wedges without piezoelectric effect (Yin, 2003c).

8. Conclusion

For all types of nondegenerate and degenerate piezoelectric materials, Eq. (4.5) gives the two-dimensional general solution in terms of the base matrix $\mathbf{Z}$ and the kernel matrix $\|f\|$ containing four arbitrary analytic functions of the complex variables $x + \mu y$. For the various classes of degenerate materials [classes (5)–(14) in the classification of Section 4], the base matrix contains higher-order eigenvectors, and the kernel matrix is block diagonal, where each diagonal block is given by Eqs. (4.1) and (4.2) for a distinct eigenvalue μ. When Eq. (4.5) is used to represent the solutions of boundary value problems, all field equations of elasticity are automatically satisfied, and the analytic functions need only be chosen in such a way as to satisfy the boundary conditions.

For nondegenerate materials, orthogonality of eigenvectors implies that $\mathbf{\Omega} = [\![\mathbf{Z}, \mathbf{Z}]\!]$ is a diagonal matrix. In the degenerate cases, $\mathbf{\Omega}$ is block diagonal and its inverse matrix is easily obtained analytically. Then $\mathbf{\Gamma} = \mathbf{Z}\langle -\mathrm{i}\mathbf{I}_4, \mathrm{i}\mathbf{I}_4\rangle \mathbf{\Omega}^{-1}\mathbf{Z}^{\mathrm{T}}$ gives the intrinsic matrix whose 4×4 submatrices $\mathbf{L}$, $\mathbf{H}$ and $\mathbf{S}$ occur in many important problems.

Although the higher-order eigenvectors require more elaborate analytical derivation than the zeroth-order eigenvectors, they do not necessarily lead to more complicated solutions of boundary value problems, because degeneracy implies a reduction in the number of distinct complex variables in the general solution. This is particularly true when the multiple eigenvalues are $\pm\mathrm{i}$. When the piezoelectric coupling is absent and the material is isotropic, 2-D Green's function of the infinite space is a linear combination of terms proportional to θ, $\sin\theta$, $\cos\theta$, $\sin 2\theta$ and $\cos 2\theta$, whereas the solutions for the materials with all distinct eigenvalues involve logarithmic functions of various complex arguments. In some analytical methods of solution, the choice of the mapping function from the domain boundary to a unit circle is also made easier if the material has a multiple eigenvalue. Hence the familiar case with all distinct eigenvalues, though apparently simple, may in fact yield more complex algebraic forms of expressions and solutions, making the analysis and computational implementation difficult compared to some degenerate cases. Suggestions have been made by some authors to avoid the analysis of degenerate materials by replacing them with nondegenerate materials having proximate eigenvalues. Aside from the questionable analytical and methodological justification (should one lay aside the entire subject of isotropic elasticity by solving the problems of isotropic materials approximately using anisotropic material models with two or more distinct pairs of proximate eigenvalues?), the suggested perturbation approach also fails to make the problem computationally simple.

The mathematical structure of the present theory exhibits complete symmetry with respect to the electric displacement vector on the one hand, and the anti-plane shearing stress vector on the other. Hence the previous solutions of pure elasticity problems with coupled in-plane and out-of-plane behavior should lead, through straightforward substitution of the variables and constitutive parameters, to the corresponding solutions of the piezoelectric problem when such coupling is replaced by electro-mechanical coupling.

It is clear that the theories of 2-D anisotropic elasticity, piezoelectricity and coupled anisotropic laminates show important common features in their essential mathematical structures, and this similarity results in formally analogous expressions of the eigensolutions, the general solution, the intrinsic matrices and Green's function. A unified formalism has been developed to encompass the various theories in a general and concise manner, including as its key results the eigenrelations, the characteristic equation for the eigenvalues, the derivative rule, orthogonality and the structure of the eigenspaces, intrinsic tensors, and Green's function of the infinite space, all given in common formal expression regardless of the context. The particular results for 2-D anisotropic elasticity, piezoelectricity theory and coupled anisotropic laminates emerge as special cases in the unified formalism. Furthermore, similar results for a new or expanded theory (for example, unsymmetric anisotropic laminates with piezoelectric effects) may be obtained likewise without repeating elaborate analysis. However, since the matrix $\mathbf{M}(\mu)$ has different dimensions and its elements have different polynomial degrees in the various theories, the number of possible types of eigenvalues and eigensolutions, as well as their detailed analytical expressions, also depend on the specific theory. The determination of all such types and their combinations constitutes the central mathematical task in obtaining the explicit expressions of the general solution and Green's functions for a particular class of materials in a specific theory. As seen in the present analysis, this task is significantly more complicated for 2-D piezoelectricity in comparison with 2-D anisotropic elasticity and the theory of coupled anisotropic laminates.

All analytical derivations contained in the present paper are easily implemented by using symbolic algebra. For materials with all distinct eigenvalues, the Appendix A lists the algorithm in *Mathematica* for obtaining the analytical expressions and numerical results of material eigenvalues, eigenvectors, the matrices $\mathbf{Z}$, $\mathbf{\Omega}$ and $\mathbf{\Gamma}$, and Green's function of the infinite space. For degenerate materials, the scheme of differentiation to generate higher-order eigenvectors from the zeroth-order eigenvectors is easily formulated. It is not difficult to modify the algorithm of the Appendix A for adoption to the various classes of degenerate materials.

Appendix A. Mathematica program for the case of distinct eigenvalues

(A) Analytical expressions

```
bb = {{b11, b12, b16, b15, b14}, {b12, b22, b26, b25, b24}, {b16, b26, b66, b56, b46},
{b15, b25, b56, b55, b45}, {b14, b24, b46, b45, b44}};
cc = {c11, c12, c16, c15, c14}, {c21, c22, c26, c25, c24};
ee = {e11, e12}, {e12, e22};
mat = Transpose[Join[Transpose[Join[bb, cc]],Transpose[Join[Transpose[cc], −ee]]]];
si = {{−s^2,0,0}, {−1,0,0}, {s,0,0}, {0,s,0}, {0,−1,0}, {0,0,s}, {0,0,−1}};
mm = Transpose[si].mat.si;
dt = Det[mm];
adj = Minors[mm,2];
j1 = {{−s,0,0}, {1,0,0}, {0,1,0}, {0,0,1}};
j3 = {{1,0,0,0,0,0,0}, {−s,0,1,0,0,0,0}, {0,0,0,1,0,0,0}, {0,0,0,0,0,1,0}};
j2 = j3.mat.si;
jj = Join[j1,j2];
vec3 = jj.adj;
ii8 = Join[Transpose[Join[0*IdentityMatrix[4],IdentityMatrix[4]]]],
      Transpose[Join[IdentityMatrix[4],0*IdentityMatrix[4]]]];
```

(*List of symbols and the corresponding equations*)

ee, cc, bb	Eqs. (I-2.10a,b,c)
mat	Eq. (I-2.19c)
si	Eq. (I-2.15b)
mm	Eq. (I-2.19b)
dt	Eq. (I-2.21)
adj	Adjoint matrix of mm
j1, j3	Eqs. (I-2.22a,b)
j2, jj	Eqs. (I-2.22c,d)
ii8	Eq. (I-2.28)

(B) Expressions for PZT-4 (the dimension of **Z**, **Ω** and **Γ** is reduced from 8 × 8 to 6 × 6)

```
ii6 = Join[Transpose[Join[0*IdentityMatrix[3],IdentityMatrix[3]]],
      Transpose[Join[IdentityMatrix[3],0*IdentityMatrix[3]]]];
bbb = bb/.{b11->8.205, b12->−3.144, b16->0, b15->0, b14->0, b22->7.495, b26->0,
   b25->0, b24->0, b66->19.3, b56->0, b46->0, b55->0, b45->0, b44->0};
ccc = cc/.{c11->0, c12->0, c16->39.4, c15->0, c14->0,
   c21->16.62, c22->23.96, c26->0, c25->0, c24->0};
eee = ee/.{e11->76.6, e12->0, e22->98.2};
mat00 = Transpose[Join[Transpose[Join[bbb,ccc]],Transpose[Join[Transpose[ccc], −eee]]]];
mat0 = mat00[[{1,2,3,6,7}, {1,2,3,6,7}]];
si0 = si[[{1,2,3,6,7}, {1,3}]];
mm0 = Transpose[si0].mat0.si0;
dt0 = Det[mm0];
adj0 = {{mm0[[2,2]], −mm0[[1,2]]}, {−mm0[[1,2]], mm0[[1,1]]}};
j10 = {{−s,0}, {1,0}, {0,1}};
j30 = {{1,0,0,0,0}, {−s,0,1,0,0}, {0,0,0,1,0}};
j20 = j30.mat0.si0;
jj0 = Join[j10,j20];
sol = NSolve[dt0==0,s];
rs = {s/.sol[[4]], s/.sol[[2]], s/.sol[[6]]};
Do[If[Im[rs[[k]]] < 0., rs[[k]] = Conjugate[rs[[k]]],],{k,3}];
Do[If[Abs[Re[rs[[k]]]] < 0.000001, rs[[k]] = I*Im[rs[[k]]],],{k,3}];
eta0 = Table[adj0/.s-> rs[[k]],{k,3}];
ind = {0,0,0};
Do[If[Abs[eta0[[i,1,1]]] > Abs[eta0[[i,2,2]]],ind[[i]] = 1,ind[[i]] = 2],{i,3}];
eta = Table[eta0[[k,ind[[k]]]],{k,3}];
vv = Table[(jj0/.s > rs[[k]]).eta[[k]],{k,3}];
zzptr = Table[vv[[k]],{k,3}];
zz = Transpose[Join[zzptr,Conjugate[zzptr]]];
omeg3 = DiagonalMatrix[Table[vv[[k]].ii6.vv[[k]],{k,3}]];
omeg6 = Join[Transpose[Join[omeg3,0*IdentityMatrix[3]]];
   Transpose[Join[0*IdentityMatrix[3],Conjugate[omeg3]]]];
iomeg6 = Inverse[omeg6];
izz = iomeg6.Transpose[zz].ii6;
gg = 2*Im[Transpose[zzptr].iomeg3.zzptr];
ll = Chop[-gg[[{1,2,3},{1,2,3}]]];
ss = Chop[gg[[{4,5,6},{1,2,3}]]];
hh = Chop[gg[[{4,5,6},{4,5,6}]]];
npi = N[Pi];
kerdiag1[t_]: = Table[Log[Cos[t] + rs[[k]]*Sin[t]],{k,3}];
kerdiag2[t_]: = Table[Log[Cos[t] + Conjugate[rs[[k]]]*Sin[t]],{k,3}];
kergrinf[t_]: = DiagonalMatrix[Join[kerdiag1[t],kerdiag2[t]]];
grinf[r_,t_]: = (0.5/npi)*Log[r]*gg.ii6 + (0.5/npi)*gg.ii6.zz.kerginf[t_].izz;
```

(* Symbols *)

mm0	$\mathbf{M}(\mu)$	Eq. (I-2.19b)
rs, zzptr	$\{\mu_\perp\}, \mathbf{Z}_\perp$	
omeg3, omeg6	$\boldsymbol{\Omega}_\perp, \boldsymbol{\Omega}$	Eqs. (3.2b,c)
iomeg6	$\boldsymbol{\Omega}^{-1}$	Eq. (3.4)
zz, izz	$\mathbf{Z}$, $\mathbf{Z}^{-1}$	Eqs. (3.2a), (3.5)
gg	$\boldsymbol{\Gamma}$	Eq. (3.15)
ll, hh, ss	$\mathbf{L}$, $\mathbf{H}$, $\mathbf{S}$	Eq. (3.22a)
grinf[r, t]	$(1/2\pi)\mathbf{G}_\infty$	Eq. (5.6)

(C) Numerical results for PZT-3

mm0 = {{7.495 + 16.156*s^2 − s^2*(3.144 8.205*s^2), 23.96 + 22.78*s^2},
{23.96 + 22.78*s^2, −98.2 − 76.60*s^2}};
rs = {1.218486708382853*I, −0.200608697306888 + 1.069879022802832*I,
0.2006086973068879 + 1.069879022802832*I};
zzptr = {{−18.92161*I, 15.52878, 9.86169, 401.8960, 348.92583*I, −174.94028*I},
{32.44963 + 21.149932*I, −13.60317 + 32.88086*I, 1.198170 + 9.778408*I,
−261.92835 ± 515.93635*I, −548.49532 − 51.242350*I, −458.73637 −781.23947*I},
{−32.44963 + 21.149932*I, −13.60317 − 32.88086*I, 1.198170 − 9.778408*I,
−261.92835 515.93635*I, 548.49532 − 51.242350*I, 458.73637 − 781.23947*I}};
omeg3 = {{−7822.66791*I, 0, 0}, {0, −6351.3916 − 23115.0640*I, 0},
{0, 0, 6351.3916 −23115.0640*I}};
iomeg3 = {{−0.0001278336*I, 0, 0}, {0, −0.0000110527 + 0.00004022485*I, 0},
{0, 0, 0.0000110527 + 0.00004022485*I}};
ll = {{0.05477005, 0, 0}, {0, 0.04298086, 0.01106335}, {0, 0.01106335, −0.008674657}};
hh = {{21.45349, 0, 0}, {0, 14.37115, 29.66584}, {0, 29.66584, −103.856287}};
ss = {{0, 0.1303836, 0.2369702}, {−0.4150223, 0, 0}, {0.9668775, 0, 0}};

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